Sequences and Series Formula Sheet

Arithmetic sequences and sums

Constant difference d	$a_n = a_1 + (n - 1)d$
Sum of first n terms	$S_n = n(a_1 + a_n)/2$
Example: 4, 7, 10, ...	$a_8 = 25, S_8 = 116$

Check the first differences before using the formula.

Geometric sequences and sums

Common ratio r	$a_n = a_1 r^{n-1}$
Finite sum, r != 1	$S_n = a_1(1 - r^n)/(1 - r)$
Finite sum, r = 1	$S_n = n a_1$
Infinite sum, |r| < 1	$S = a_1/(1 - r)$

The nth-term formula uses a_1 for n = 1, including when r = 0.

Common sums from 1 to n

First powers	$\sum k = n(n + 1)/2$
Squares	$\sum k^2 = n(n + 1)(2n + 1)/6$
Cubes	$\sum k^3 = [n(n + 1)/2]^2$

For an index starting elsewhere, subtract the unwanted initial terms.

Convergence tests

Term limit nonzero or absent	**series diverges**

p-series, sum 1/n^p

converges exactly when p > 1

Ratio L = lim |a_next/a_n|

L < 1 converges absolutely

Ratio L > 1	**diverges. L = 1 is inconclusive.**

A zero term limit alone does not show convergence.

Comparison and alternating series

Direct comparison	$0 \le a_n \le b_n$ **eventually**
Limit comparison	$0 < \lim(a_n/b_n) < \infty$
Alternating series	**magnitudes decrease to 0**

Alternating error

|error| ≤ next term magnitude

For comparison, a convergent upper bound or a divergent lower bound decides.
With a positive finite ratio limit, both positive series behave alike.

Taylor series and radius

Taylor coefficient at a	$c_k = D^k f(a)/k!$
Power series	$\sum c_k (x - a)^k$
Inside radius	$\|x - a\| < R$
Remainder bound	$\|R_n\| \le M\|x - a\|^{n+1}/(n + 1)!$

D^k means the kth derivative. Check that the Taylor remainder tends to zero.
Test endpoints separately. M bounds the size of derivative n + 1 between a and x.

Maclaurin series to recognize

Exponential	$e^x = 1 + x + x^2/2! + x^3/3! + \dots$
Sine	$\sin x = x - x^3/3! + x^5/5! - \dots$
Cosine	$\cos x = 1 - x^2/2! + x^4/4! - \dots$
Geometric	$1/(1 - x) = 1 + x + x^2 + \dots$

The first three converge for all real x. The last requires |x| < 1.

Root and integral tests

Root-test input	$L = \lim \|a_n\|^{1/n}$
Root-test decision	**L < 1 converges absolutely**

Root-test limits

L > 1 diverges, L = 1 inconclusive

Integral test	$\sum f(n)$ **and** $\int f$ **share convergence**

For the integral test, require f positive, continuous and decreasing eventually.

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Binomial and arctangent series

Binomial coefficient in series

$$c_k = r(r-1)\dots(r-k+1)/k!$$

General binomial $(1+x)^r = 1 + \sum c_k\, x^k$

Arctangent $\operatorname{atan} x = x - x^3/3 + x^5/5 - \dots$

Example: repeating decimal

$$0.2727\ldots = 0.27/(1-0.01) = 3/11$$

The stated power series hold for $|x| < 1$. Check endpoints separately.