Derivative Rules: Expanded

Power and linearity rules

Constant	$(c)' = 0$
Power	$(x^n)' = n \times x^{n-1}$
Constant multiple	$(cf)' = cf'$
Sum or difference	$(f \pm g)' = f' \pm g'$

For the power rule, use positive x for general real n. Positive integer powers also work for negative x and zero.

Product rule

Two factors	$(fg)' = f'g + fg'$
Example	$(x^2 \sin x)' = 2x \sin x + x^2 \cos x$

Differentiate one factor at a time. Both original functions must be differentiable at the point.

Quotient rule

A ratio	$(f/g)' = (f'g - fg') / g^2$
Example	$(x/(x + 1))' = 1/(x + 1)^2$

Keep the subtraction in order. Use the rule only where both functions are differentiable and g is nonzero.

Chain rule

A composition	$(f(g(x)))' = f'(g(x)) \times g'(x)$
Outer function	$f(u) = u^3$
Inner function	$g(x) = 2x - 1$

Differentiate the outer function at the inner value, then multiply by the derivative of the inner function.

Common trig derivatives

Sine	$(\sin x)' = \cos x$
Cosine	$(\cos x)' = -\sin x$
Tangent	$(\tan x)' = \sec^2 x$

Angles are in radians. Tangent is differentiable where cos x is nonzero.

Exponential and log derivatives

Natural exponential	$(\exp(x))' = \exp(x)$
Exponential base a	$(a^x)' = a^x \ln(a)$
Natural logarithm	$(\ln x)' = 1/x$

exp(x) means e to the power x. Use a > 0 for a^x and x > 0 for ln x.

Example: Polynomial and slope

Function	$f(x) = 2x^3 - 5x + 4$
Differentiate term by term	$f'(x) = 6x^2 - 5$
Slope at x = 2	$f'(2) = 24 - 5 = 19$

A derivative at a point is the slope of the tangent line there.

Example: A nested power

Function	$h(x) = (2x - 1)^3$
Outer derivative at the inside	$3(2x - 1)^2$
Inner derivative	2
Multiply	$h'(x) = 6(2x - 1)^2$

The extra factor of 2 comes from the inner linear function.

Derivative Rules: Expanded

Implicit and inverse derivatives

Implicit example	$x^2 + y^2 = 25$
Differentiate both sides	$2x + 2yy' = 0$
If y != 0	$y' = -x/y$
Inverse g of f	$g'(x) = 1/f'(g(x))$

For an inverse derivative, work on an invertible branch with nonzero f′.

Tangent and extrema checks

Linearization at a	$L(x) = f(a) + f'(a)(x - a)$
Mean value theorem	$f'(c) = [f(b) - f(a)]/(b - a)$
Critical point, interior	$f'(c) = 0$ or $f'(c)$ undefined

For MVT, f is continuous on [a,b] and differentiable on (a,b).
If $f'(c) = 0$, $f''(c) > 0$ gives a local min and $f''(c) < 0$ a local max.

Inverse trig derivatives

Arcsine, $\|x\| < 1$	$(\arcsin x)' = 1/\sqrt{(1 - x^2)}$
Arccosine, $\|x\| < 1$	$(\arccos x)' = -1/\sqrt{(1 - x^2)}$
Arctangent	$(\arctan x)' = 1/(1 + x^2)$
Arcsecant, $\|x\| > 1$	$(\operatorname{arcsec} x)' = 1/(\|x\|\sqrt{(x^2 - 1)})$

Use principal inverse functions. For a composed input u(x), multiply by u′(x).

Hyperbolic derivatives

Hyperbolic sine	$(\sinh x)' = \cosh x$
Hyperbolic cosine	$(\cosh x)' = \sinh x$
Hyperbolic tangent	$(\tanh x)' = \operatorname{sech}^2(x)$

Log differentiation example

$x > 0$, $(x^x)' = x^x(1 + \ln x)$

Hyperbolic functions differ from circular trig. The derivative of cosh has a positive sign.