

Original olympiad reasoning practice - Edition 1

Student paper

Name: _____ Date: _____

Write a proof or a complete justified solution, rather than only a numerical answer. Define any new variables you use.

The problems are grouped by increasing demand: four warm-ups, four core arguments, and four stretch proofs. No time limit is prescribed.

Make an independent attempt before using hint 1, and use hint 2 only if still stuck. The teacher edition includes full model arguments.

1. Prove that $n^3 - n$ is divisible by 6 for every integer n .

2. Let x and y be positive real numbers with $xy = 1$. Prove that $x + y \geq 2$, and determine exactly when equality holds.

Make an independent attempt before using hint 1, and use hint 2 only if still stuck. The teacher edition includes full model arguments.

5. Find all positive integer pairs (x, y) with $x \leq y$ that satisfy $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$. Prove that your list is complete.

6. Real numbers a , b , and c satisfy $a + b + c = 0$. Prove that $a^3 + b^3 + c^3 = 3abc$. Deduce that if $a^3 + b^3 + c^3 = 0$ as well, at least one of a , b , c is zero.

Make an independent attempt before using hint 1, and use hint 2 only if still stuck. The teacher edition includes full model arguments.

7. Triangle ABC is right-angled at A. D is the foot of the perpendicular from A to hypotenuse BC. Write $BD = p$, $DC = q$, and $AD = h$. Prove $h^2 = pq$.

8. The numbers 1 through 8 are arranged around a circle, each used once. Prove that some two adjacent numbers have sum at least 10. The last and first positions are adjacent.

Make an independent attempt before using hint 1, and use hint 2 only if still stuck. The teacher edition includes full model arguments.

9. Prove that there are no positive integers a, b, c satisfying $a^2 + b^2 = 3c^2$.

10. For positive real numbers a , b , and c , prove $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$. Determine all equality cases.

Make an independent attempt before using hint 1, and use hint 2 only if still stuck. The teacher edition includes full model arguments.

11. In an acute triangle ABC , let H be the common intersection of its three altitudes. Let H' be the reflection of H across line BC . Prove that H' lies on the circle through A , B , and C .

12. A 2 by n rectangle is tiled by 1 by 2 dominoes, which may be rotated. Let T_n be the number of tilings, and count the empty 2 by 0 board as having one tiling. Prove that T_n is even exactly when n leaves remainder 2 on division by 3.

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Worked solutions

Use after attempting the student paper. Alternative valid methods are welcome.

1. Prove that $n^3 - n$ is divisible by 6 for every integer n .

Answer: 6 divides $n^3 - n$ for every integer n .

Factor $n^3 - n = (n - 1)n(n + 1)$. Among any three consecutive integers, exactly one is a multiple of 3, and at least one is even. Hence their product has a factor 3 and a factor 2. Since 2 and 3 are relatively prime, it is divisible by 6. This also covers negative integers and zero.

2. Let x and y be positive real numbers with $xy = 1$. Prove that $x + y \geq 2$, and determine exactly when equality holds.

Answer: $x + y \geq 2$, with equality exactly at $x = y = 1$.

We have $(x + y)^2 = (x - y)^2 + 4xy = (x - y)^2 + 4 \geq 4$. Since $x + y$ is positive, taking square roots gives $x + y \geq 2$.

Equality holds exactly when $(x - y)^2 = 0$, or $x = y$. Together with $xy = 1$ and positivity, this forces $x = y = 1$. Those values attain equality.

3. In a nondegenerate triangle ABC , $AB = AC$. Let D be the midpoint of BC . Prove that AD is perpendicular to BC and that AD bisects angle BAC .

Answer: AD is perpendicular to BC and angle BAD equals angle DAC .

Triangles ABD and ACD have $AB = AC$, $BD = DC$ because D is the midpoint, and AD as a common side. They are congruent by the side-side-side criterion. Thus angle BAD equals angle DAC , so AD bisects angle BAC .

Congruence also gives angle $BDA = \text{angle } ADC$. Because B, D, C are collinear, these two angles sum to 180 degrees. Each is therefore 90 degrees, proving AD perpendicular to BC .

4. Prove that among any five integers there are two whose difference is divisible by 4. The integers need not be distinct.

Answer: Two of the five integers have difference divisible by 4.

Every integer has one of the four remainders 0, 1, 2, 3 on division by 4. With five integers in four classes, the pigeonhole principle gives two in the same class. Write them as $4a + r$ and $4b + r$. Their difference is $4(a - b)$, which is divisible by 4. This includes the possibility of equal integers, whose difference is 0.

5. Find all positive integer pairs (x, y) with $x \leq y$ that satisfy $1/x + 1/y = 1/6$. Prove that your list is complete.

Answer: (7,42), (8,24), (9,18), (10,15), and (12,12).

Multiplying by $6xy$ gives $xy = 6x + 6y$, or $(x - 6)(y - 6) = 36$. Also $1/x = 1/6 - 1/y < 1/6$, so $x > 6$; similarly $y > 6$. Thus both new factors are positive integers.

Since $x \leq y$, the factor $x - 6$ is at most $\sqrt{36} = 6$. The positive divisors of 36 up to 6 are 1, 2, 3, 4, and 6, giving factor pairs (1,36), (2,18), (3,12), (4,9), and (6,6).

Adding 6 to each entry gives exactly (7,42), (8,24), (9,18), (10,15), and (12,12). Each pair satisfies the product equation and hence the original equation, so the list is complete.

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Worked solutions

Use after attempting the student paper. Alternative valid methods are welcome.

6. Real numbers a , b , and c satisfy $a + b + c = 0$. Prove that $a^3 + b^3 + c^3 = 3abc$. Deduce that if $a^3 + b^3 + c^3 = 0$ as well, at least one of a , b , c is zero.

Answer: The cube sum equals $3abc$; a zero cube sum forces $abc = 0$.

Since $c = -(a + b)$, we have $a^3 + b^3 + c^3 = a^3 + b^3 - (a + b)^3 = -3a^2b - 3ab^2 = -3ab(a + b) = 3abc$.

If the cube sum is also zero, then $3abc = 0$, so $abc = 0$. A product of real numbers is zero only when at least one factor is zero. Hence at least one of a , b , c is zero.

7. Triangle ABC is right-angled at A . D is the foot of the perpendicular from A to hypotenuse BC . Write $BD = p$, $DC = q$, and $AD = h$. Prove $h^2 = pq$.

Answer: The altitude satisfies $h^2 = pq$.

Because the original triangle is nondegenerate and right-angled at A , the foot D lies between B and C and p , q , h are positive. Triangles ABD and CAD are right-angled at D .

Angle ABD is angle ABC . In triangle ACD , angle $CAD = 90^\circ - \text{angle } ACD = 90^\circ - \text{angle } ACB = \text{angle } ABC$. Thus triangles ABD and CAD are similar by two angles, with B corresponding to A , A to C , and D to D .

The corresponding sides give $BD/AD = AD/DC$, or $p/h = h/q$. Multiplying by hq yields $h^2 = pq$.

8. The numbers 1 through 8 are arranged around a circle, each used once. Prove that some two adjacent numbers have sum at least 10. The last and first positions are adjacent.

Answer: At least one adjacent pair has sum at least 10.

Each number appears in exactly two adjacent-pair sums, so the total of those eight sums is $2(1 + 2 + \dots + 8) = 72$. Their average is 9.

Suppose no adjacent sum is at least 10. Because the sums are integers, they are all at most 9. Their total 72 then forces every sum to be exactly 9. If three successive entries are a , b , c , we get $a + b = b + c = 9$, hence $a = c$. This repeats a number in distinct positions, contrary to using each number once. Therefore some adjacent sum is at least 10.

9. Prove that there are no positive integers a , b , c satisfying $a^2 + b^2 = 3c^2$.

Answer: No positive integer triple satisfies the equation.

Suppose a positive solution exists, and choose one with c as small as possible. Modulo 3, a square is 0 or 1. Since $a^2 + b^2$ is 0 modulo 3, both squares must be 0 modulo 3. Thus $a = 3u$ and $b = 3v$ for positive integers u , v .

Substitution gives $9(u^2 + v^2) = 3c^2$, or $c^2 = 3(u^2 + v^2)$. Hence 3 divides c^2 and therefore c ; write $c = 3w$. The original equation now reduces to $u^2 + v^2 = 3w^2$.

This is another positive solution with $w = c/3 < c$, contradicting the minimal choice of c . The assumption of a solution is impossible.

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Worked solutions

Use after attempting the student paper. Alternative valid methods are welcome.

10. For positive real numbers a , b , and c , prove $a/(b+c) + b/(c+a) + c/(a+b) \geq 3/2$. Determine all equality cases.

Answer: The sum is at least $3/2$, with equality exactly when $a = b = c$.

For any positive u, v, w , expand $(u+v+w)(1/u + 1/v + 1/w)$. The three diagonal terms sum to 3, and each of the three pairs $u/v + v/u$, $v/w + w/v$, $w/u + u/w$ is at least 2, since for example $u/v + v/u - 2 = (u-v)^2/(uv) \geq 0$. Therefore the product is at least 9, with equality exactly when $u = v = w$.

Set $s = a + b + c$ and take $u = s - a$, $v = s - b$, $w = s - c$. These are positive and sum to $2s$, so $1/(s-a) + 1/(s-b) + 1/(s-c) \geq 9/(2s)$.

Now $a/(s-a) = s/(s-a) - 1$, and similarly for the other terms. Their sum is at least $s \times 9/(2s) - 3 = 3/2$. Equality requires $s-a = s-b = s-c$, equivalent to $a = b = c$; any positive equal triple attains it.

11. In an acute triangle ABC , let H be the common intersection of its three altitudes. Let H' be the reflection of H across line BC . Prove that H' lies on the circle through A, B , and C .

Answer: The reflection H' is on the circumcircle of triangle ABC .

Let angle BAC be α . Because the triangle is acute, H is inside it. In triangle BHC , angle $HBC = 90^\circ - \text{angle } ACB$, and angle $HCB = 90^\circ - \text{angle } ABC$. Thus angle $BHC = \text{angle } ABC + \text{angle } ACB = 180^\circ - \alpha$.

Reflection in BC fixes B and C and preserves angles, so angle $BH'C = \text{angle } BHC = 180^\circ - \alpha$. H' is on the opposite side of BC from A , since H is on the same side as A .

The simple quadrilateral $A-B-H'-C$ has opposite angles BAC and $BH'C$ summing to 180° . By the converse of the cyclic-quadrilateral angle theorem, its four vertices lie on a circle. Thus H' lies on the circle through A, B , and C .

12. A 2 by n rectangle is tiled by 1 by 2 dominoes, which may be rotated. Let T_n be the number of tilings, and count the empty 2 by 0 board as having one tiling. Prove that T_n is even exactly when n leaves remainder 2 on division by 3 .

Answer: T_n is even if and only if $n \equiv 2 \pmod{3}$.

We have $T_0 = 1$ and $T_1 = 1$. For $n \geq 2$, the leftmost column is either covered by one vertical domino, leaving a 2 by $(n-1)$ board, or by the ends of two horizontal dominoes. In the second case both horizontal dominoes are forced, leaving a 2 by $(n-2)$ board. These cases are disjoint and exhaustive, so $T_n = T_{n-1} + T_{n-2}$.

Modulo 2, this gives $T_{n+3} = T_{n+2} + T_{n+1} = T_n + 2T_{n+1}$, which has the same parity as T_n for every $n \geq 0$. Thus the parity has period 3. The first three values are 1, 1, 2, with parities odd, odd, even. Therefore T_n is even exactly for $n \equiv 2 \pmod{3}$.