Original olympiad reasoning practice - Edition 1

Hint sheet

Try independently first. Read hint 1 before hint 2.

1. Prove that $n^3 - n$ is divisible by 6 for every integer n.

Hint 1: Factor the expression.

Hint 2: Among three consecutive integers, one is divisible by 3 and at least one is even.

2. Let x and y be positive real numbers with $xy = 1$. Prove that $x + y >= 2$, and determine exactly when equality holds.

Hint 1: A square is always nonnegative.

Hint 2: Expand $(x - y)^2$ and use $xy = 1$.

3. In a nondegenerate triangle ABC, AB = AC. Let D be the midpoint of BC. Prove that AD is perpendicular to BC and that AD bisects angle BAC.

Hint 1: Compare triangles ABD and ACD.

Hint 2: Use three equal pairs of sides, then examine the two angles at D.

4. Prove that among any five integers there are two whose difference is divisible by 4. The integers need not be distinct.

Hint 1: Classify the integers by their remainders modulo 4.

Hint 2: There are four remainder classes but five integers.

5. Find all positive integer pairs (x, y) with $x <= y$ that satisfy $1/x + 1/y = 1/6$. Prove that your list is complete.

Hint 1: Multiply by 6xy and complete a product.

Hint 2: The equation is equivalent to $(x - 6)(y - 6) = 36$. Explain why both factors are positive.

6. Real numbers a, b, and c satisfy $a + b + c = 0$. Prove that $a^3 + b^3 + c^3 = 3abc$. Deduce that if $a^3 + b^3 + c^3 = 0$ as well, at least one of a, b, c is zero.

Hint 1: Substitute $c = -(a + b)$.

Hint 2: Expand $a^3 + b^3 - (a + b)^3$ and factor.

7. Triangle ABC is right-angled at A. D is the foot of the perpendicular from A to hypotenuse BC. Write BD = p, DC = q, and AD = h. Prove $h^2 = pq$.

Hint 1: Compare the two smaller right triangles ABD and CAD.

Hint 2: Show they are similar with BD corresponding to AD and AD corresponding to DC.

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8. The numbers 1 through 8 are arranged around a circle, each used once. Prove that some two adjacent numbers have sum at least 10. The last and first positions are adjacent.

Hint 1: Add all eight adjacent-pair sums.

Hint 2: If every sum were at most 9, their average would force all of them to be exactly 9. What would that imply for alternating positions?

9. Prove that there are no positive integers a, b, c satisfying $a^2 + b^2 = 3c^2$.

Hint 1: Squares have remainder 0 or 1 modulo 3.

Hint 2: Show that a, b, and c would all be divisible by 3, producing a smaller positive solution.

10. For positive real numbers a, b, and c, prove $a/(b + c) + b/(c + a) + c/(a + b) >= 3/2$. Determine all equality cases.

Hint 1: For positive u and v, $u/v + v/u >= 2$ follows from $(u - v)^2 >= 0$.

Hint 2: Apply the three-variable reciprocal inequality to $u = b + c$, $v = c + a$, $w = a + b$.

11. In an acute triangle ABC, let H be the common intersection of its three altitudes. Let H' be the reflection of H across line BC. Prove that H' lies on the circle through A, B, and C.

Hint 1: Use BH perpendicular to AC and CH perpendicular to AB to express angle BHC in terms of angle BAC.

Hint 2: Reflection preserves angle BHC. A and H' are on opposite sides of BC, so use the supplementary opposite-angle test for a cyclic quadrilateral.

12. A 2 by n rectangle is tiled by 1 by 2 dominoes, which may be rotated. Let T_n be the number of tilings, and count the empty 2 by 0 board as having one tiling. Prove that T_n is even exactly when n leaves remainder 2 on division by 3.

Hint 1: Classify a tiling by the domino or pair of dominoes covering its left edge.

Hint 2: Derive $T_n = T_{(n-1)} + T_{(n-2)}$, then examine the recurrence modulo 2.