

Math Sheet

Name _____ Score _____ Teacher _____ Date _____

Taylor Series Approximations and Error

Use standard Taylor or Maclaurin series to represent a function, approximate a nearby value, and distinguish the exact target from the polynomial approximation and its error bound.

1. For $f(x)=1/(1-x)$ centered at $a=0$, Write the order-4 Taylor polynomial and the infinite-series representation, state its convergence interval, and verify that $x=1/10$ is inside it. Use the lagrange method for the error statement.

2. For $f(x)=\cos(x)$ centered at $a=0$, Write the order-4 Taylor polynomial and the infinite-series representation, state its convergence interval, and verify that $x=4/5$ is inside it. Use the lagrange method for the error statement.

Taylor Series Approximations and Error

Use standard Taylor or Maclaurin series to represent a function, approximate a nearby value, and distinguish the exact target from the polynomial approximation and its error bound.

3. For $f(x)=\ln(1+x)$ centered at $a=0$, Use the lagrange remainder method to bound the error at the exact point $x=2/5$. Use the lagrange method for the error statement.

4. For $f(x)=e^x$ centered at $a=0$, Write the order-4 Taylor polynomial and the infinite-series representation, state its convergence interval, and verify that $x=9/20$ is inside it. Use the lagrange method for the error statement.

Taylor Series Approximations and Error

Use standard Taylor or Maclaurin series to represent a function, approximate a nearby value, and distinguish the exact target from the polynomial approximation and its error bound.

5. For $f(x)=\cos(x)$ centered at $a=0$, Use the order-4 polynomial to approximate the exact target $f(2/5)=\cos(2/5)$. Use the alternating method for the error statement.

6. For $f(x)=\sin(x)$ centered at $a=0$, Use the lagrange remainder method to bound the error at the exact point $x=2/5$. Use the lagrange method for the error statement.

Taylor Series Approximations and Error

Worked Answer Key

Use standard Taylor or Maclaurin series to represent a function, approximate a nearby value, and distinguish the exact target from the polynomial approximation and its error bound.

1. For $f(x)=1/(1-x)$ centered at $a=0$, Write the order-4 Taylor polynomial and the infinite-series representation, state its convergence interval, and verify that $x=1/10$ is inside it. Use the lagrange method for the error statement.

Answer: $P_4(x)=1+x+x^2+x^3+x^4$; $f(x)=\sum_{k=0}^{\infty} x^k$, $|x|<1$; $x=1/10$ is admitted.

- 1. $P_4(x)=1+x+x^2+x^3+x^4$.
- 2. The full representation is $f(x)=\sum_{k=0}^{\infty} x^k$, converging for $|x|<1$; $h=1/10$ satisfies this condition.
- 3. Substitute the exact displacement $h=1/10$: $P_4(1/10)$ is approximately 1.1111 while the exact target is $1/(1-1/10)$.
- 4. On the interval from a to $a+h$, $|f^{(5)}(t)|\leq M$ with $M=5/(9/10)^6$. Lagrange gives $|R_4|\leq 5/(9/10)^6 * (1/10)^5/5!$.
- 5. Using the unrounded polynomial value, the actual error is approximately 0.00011 and is no greater than the exact bound $5/(9/10)^6 * (1/10)^5/5!$.

2. For $f(x)=\cos(x)$ centered at $a=0$, Write the order-4 Taylor polynomial and the infinite-series representation, state its convergence interval, and verify that $x=4/5$ is inside it. Use the lagrange method for the error statement.

Answer: $P_4(x)=1-1/2!x^2+1/4!x^4$; $f(x)=\sum_{j=0}^{\infty} (-1)^j x^{(2j)}/(2j)!$, all real x ; $x=4/5$ is admitted.

- 1. $P_4(x)=1-1/2!x^2+1/4!x^4$.
- 2. The full representation is $f(x)=\sum_{j=0}^{\infty} (-1)^j x^{(2j)}/(2j)!$, converging for all real x ; $h=4/5$ satisfies this condition.
- 3. Substitute the exact displacement $h=4/5$: $P_4(4/5)$ is approximately 0.697067 while the exact target is $\cos(4/5)$.
- 4. On the interval from a to $a+h$, $|f^{(5)}(t)|\leq M$ with $M=1$. Lagrange gives $|R_4|\leq 1 * (4/5)^5/5!$.
- 5. Using the unrounded polynomial value, the actual error is approximately 0.00036 and is no greater than the exact bound $1 * (4/5)^5/5!$.

Taylor Series Approximations and Error

Worked Answer Key

Use standard Taylor or Maclaurin series to represent a function, approximate a nearby value, and distinguish the exact target from the polynomial approximation and its error bound.

3.

For $f(x)=\ln(1+x)$ centered at $a=0$, Use the lagrange remainder method to bound the error at the exact point $x=2/5$. Use the lagrange method for the error statement.

Answer: Exact target $\ln(1+2/5)$; $P_4(2/5)$ is approximately 0.334933; $|R_4| \leq 4! \cdot ((2/5))^5/5!$.

- $P_4(x)=1/x - 1/2x^2 + 1/3x^3 - 1/4x^4$.
- The full representation is $f(x)=\sum_{k=1}^{\infty} (-1)^{(k+1)}x^k/k$, converging for $-1 < x \leq 1$; $h=2/5$ satisfies this condition.
- Substitute the exact displacement $h=2/5$: $P_4(2/5)$ is approximately 0.334933 while the exact target is $\ln(1+2/5)$.
- On the interval from a to $a+h$, $|f^{(5)}(t)| \leq M$ with $M=4!$. Lagrange gives $|R_4| \leq 4! \cdot ((2/5))^5/5!$.
- Using the unrounded polynomial value, the actual error is approximately 0.001539 and is no greater than the exact bound $4! \cdot ((2/5))^5/5!$.

4.

For $f(x)=e^x$ centered at $a=0$, Write the order-4 Taylor polynomial and the infinite-series representation, state its convergence interval, and verify that $x=9/20$ is inside it. Use the lagrange method for the error statement.

Answer: $P_4(x)=1 + 1/1!x + 1/2!x^2 + 1/3!x^3 + 1/4!x^4$; $f(x)=\sum_{k=0}^{\infty} x^k/k!$, all real x ; $x=9/20$ is admitted.

- $P_4(x)=1 + 1/1!x + 1/2!x^2 + 1/3!x^3 + 1/4!x^4$.
- The full representation is $f(x)=\sum_{k=0}^{\infty} x^k/k!$, converging for all real x ; $h=9/20$ satisfies this condition.
- Substitute the exact displacement $h=9/20$: $P_4(9/20)$ is approximately 1.568146 while the exact target is $e^{(9/20)}$.
- On the interval from a to $a+h$, $|f^{(5)}(t)| \leq M$ with $M=e^{9/20}$. Lagrange gives $|R_4| \leq e^{9/20} \cdot ((9/20))^5/5!$.
- Using the unrounded polynomial value, the actual error is approximately 0.000166 and is no greater than the exact bound $e^{9/20} \cdot ((9/20))^5/5!$.

Taylor Series Approximations and Error

Worked Answer Key

Use standard Taylor or Maclaurin series to represent a function, approximate a nearby value, and distinguish the exact target from the polynomial approximation and its error bound.

5. For $f(x)=\cos(x)$ centered at $a=0$, Use the order-4 polynomial to approximate the exact target $f(2/5)=\cos(2/5)$. Use the alternating method for the error statement.

Answer: Exact target $\cos(2/5)$; $P_4(2/5)$ is approximately 0.921067; $|R_4| \leq 1/6! * (2/5)^6$.

- 1. $P_4(x)=1 - 1/2!x^2 + 1/4!x^4$.
- 2. The full representation is $f(x)=\sum_{j=0}^{\infty} (-1)^j x^{(2j)}/(2j)!$, converging for all real x ; $h=2/5$ satisfies this condition.
- 3. Substitute the exact displacement $h=2/5$: $P_4(2/5)$ is approximately 0.921067 while the exact target is $\cos(2/5)$.
- 4. At $h=2/5$, the nonzero terms alternate, their magnitudes decrease, and they approach 0. Thus the alternating remainder is at most the first omitted magnitude $1/6! * (2/5)^6$.
- 5. Using the unrounded polynomial value, the actual error is approximately 0.000006 and is no greater than the exact bound $1/6! * (2/5)^6$.

6. For $f(x)=\sin(x)$ centered at $a=0$, Use the lagrange remainder method to bound the error at the exact point $x=2/5$. Use the lagrange method for the error statement.

Answer: Exact target $\sin(2/5)$; $P_4(2/5)$ is approximately 0.389333; $|R_4| \leq 1 * (2/5)^5/5!$.

- 1. $P_4(x)=1/1!x - 1/3!x^3$.
- 2. The full representation is $f(x)=\sum_{j=0}^{\infty} (-1)^j x^{(2j+1)}/(2j+1)!$, converging for all real x ; $h=2/5$ satisfies this condition.
- 3. Substitute the exact displacement $h=2/5$: $P_4(2/5)$ is approximately 0.389333 while the exact target is $\sin(2/5)$.
- 4. On the interval from a to $a+h$, $|f^{(5)}(t)| \leq M$ with $M=1$. Lagrange gives $|R_4| \leq 1 * (2/5)^5/5!$.
- 5. Using the unrounded polynomial value, the actual error is approximately 0.000085 and is no greater than the exact bound $1 * (2/5)^5/5!$.