

First-Order Linear and Logistic Equations

For linear equations, show the integrating factor. For logistic equations, give the solution formula, equilibria, units, and long-term behavior.

1. Let $P(t)$ be a population in organisms, with t in years. Solve $P' = 1/2P(1-P/180)$, $P(0) = 30$; state both equilibria and long-term behavior.

2. A population $P(t)$, measured in organisms with t in years, follows $P' = 1/4P(1-P/180)$ and $P(0) = 90$. Find $P(t)$, both equilibria, and the long-term population.

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For linear equations, show the integrating factor. For logistic equations, give the solution formula, equilibria, units, and long-term behavior.

3. A concentration $C(t)$, in mg/L, satisfies $C' + 3C = 12$, $C(0) = 7$. Find the concentration and its long-term value.

4. Solve $y' + 3y = 9$, $y(1) = 5$. Use an integrating factor and state the maximal interval.

First-Order Linear and Logistic Equations
Worked Answer Key

For linear equations, show the integrating factor. For logistic equations, give the solution formula, equilibria, units, and long-term behavior.

1. Let $P(t)$ be a population in organisms, with t in years. Solve $P' = \frac{1}{2}P(1-P/180)$, $P(0) = 30$; state both equilibria and long-term behavior.

Answer: $P(t) = \frac{180}{1 + 5e^{-(1/2)t}}$; equilibria 0 and 180; limit 180 organisms.

- Equilibria solve $\frac{1}{2}P(1-P/180) = 0$: $P = 0$ and $P = 180$.
- For a non-equilibrium branch, separate: $dP/[P(1-P/180)] = \frac{1}{2} dt$.
- Partial fractions integrate to $\ln|P/(180-P)| = (1/2)t + C$.
- The logistic form is $P(t) = \frac{180}{1 + Ae^{-(1/2)t}}$; $P(0) = 30$ gives $A = (180-30)/30 = 5$.
- Because $r > 0$ and $P_0 > 0$, $P(t)$ tends to the carrying capacity 180 organisms as t tends to infinity.

2. A population $P(t)$, measured in organisms with t in years, follows $P' = \frac{1}{4}P(1-P/180)$ and $P(0) = 90$. Find $P(t)$, both equilibria, and the long-term population.

Answer: $P(t) = \frac{180}{1 + 1e^{-(1/4)t}}$; equilibria 0 and 180; limit 180 organisms.

- Equilibria solve $\frac{1}{4}P(1-P/180) = 0$: $P = 0$ and $P = 180$.
- For a non-equilibrium branch, separate: $dP/[P(1-P/180)] = \frac{1}{4} dt$.
- Partial fractions integrate to $\ln|P/(180-P)| = (1/4)t + C$.
- The logistic form is $P(t) = \frac{180}{1 + Ae^{-(1/4)t}}$; $P(0) = 90$ gives $A = (180-90)/90 = 1$.
- Because $r > 0$ and $P_0 > 0$, $P(t)$ tends to the carrying capacity 180 organisms as t tends to infinity.

First-Order Linear and Logistic Equations
Worked Answer Key

For linear equations, show the integrating factor. For logistic equations, give the solution formula, equilibria, units, and long-term behavior.

3. A concentration $C(t)$, in mg/L, satisfies $C' + 3C = 12$, $C(0) = 7$. Find the concentration and its long-term value.

Answer: $C = 4 + 3e^{-3(t-0)}$; limit $C = 4$ mg/L.

- Standard form is $C' + 3C = 12$; the integrating factor is $\mu(t) = e^{3t}$.
- Multiplying gives $(e^{3t}C)' = 12e^{3t}$.
- Integrate: $C = 4 + Ae^{-3t}$. Apply $C(0) = 7$.
- Thus $A = 3e^{3(0)}$ and $C = 4 + 3e^{-3(t-0)}$.
- The exponential term tends to 0, so the long-term concentration is 4 mg/L (mg/L after t hours).

4. Solve $y' + 3y = 9$, $y(1) = 5$. Use an integrating factor and state the maximal interval.

Answer: $y = 3 + 2e^{-3(x-1)}$; maximal interval $(-\infty, \infty)$.

- Standard form is $y' + 3y = 9$; the integrating factor is $\mu(x) = e^{3x}$.
- Multiplying gives $(e^{3x}y)' = 9e^{3x}$.
- Integrate: $y = 3 + Ae^{-3x}$. Apply $y(1) = 5$.
- Thus $A = 2e^3$ and $y = 3 + 2e^{-3(x-1)}$.
- All coefficients are continuous on $\mathbb{R}$, so the maximal interval is $(-\infty, \infty)$.